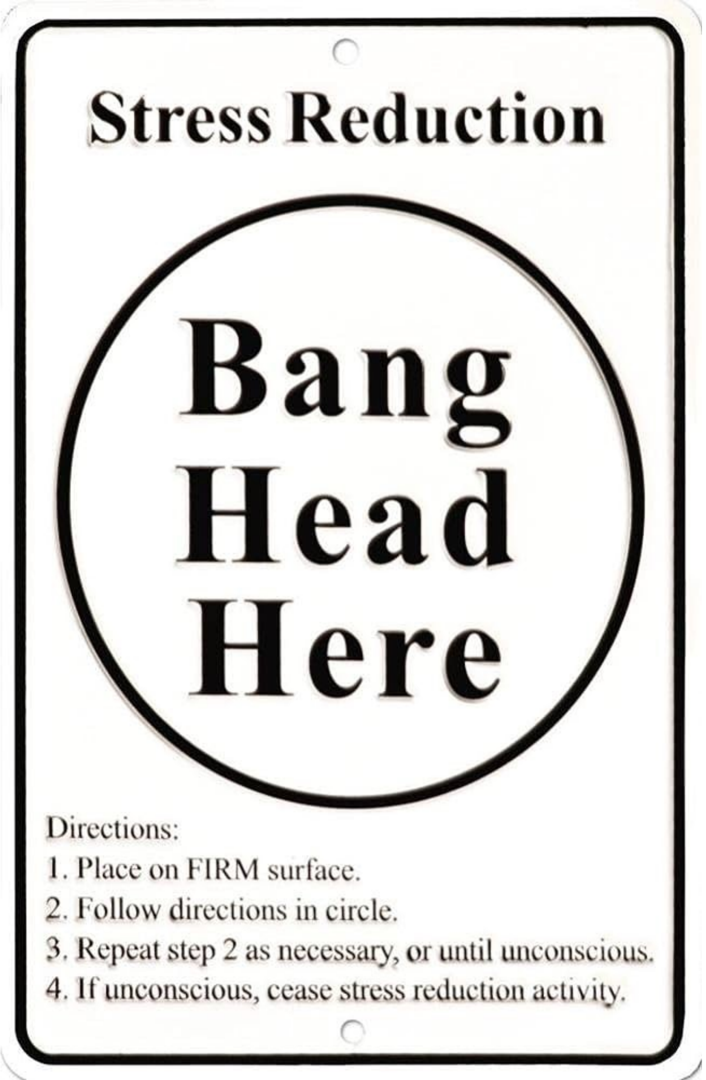


Georg Fantner

Week 10: Transformation of stresses and strains part 2

1. Mohr's circle of stress
2. Principal stresses in 3D

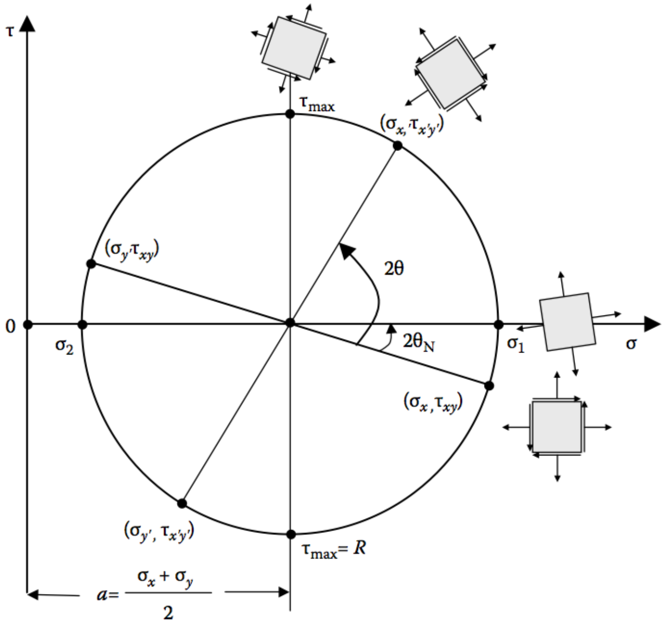
Mohr's circle of strain in 2D

$$(\varepsilon_{av}, 0) = \left(\frac{\varepsilon_x - \varepsilon_y}{2}, 0\right)$$

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\left(\frac{\gamma_{xy}}{2}\right)_{max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$(\varepsilon_{x'})^{max}_{min} = \varepsilon_1 \& \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$



Transformation of plane stress & strain in 2D

Summary

	Plane stress	Plane strain
max normal	$(\sigma_{x'})^{max}_{min} = \sigma_{1\&2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$	$(\varepsilon_{x'})^{max}_{min} = \varepsilon_{1\&2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$
max shear	$(\tau_{xy})^{max}_{min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$	$\left(\frac{\gamma_{xy}}{2}\right)_{max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$
Angle max normal	$\theta_N = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}\right)$	$\theta_N = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}\right)$
Angle max shear	$\theta_S = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}\right)$	$\theta_N = \frac{1}{2}\tan^{-1}\left(-\frac{\varepsilon_{xx} - \varepsilon_{yy}}{\gamma_{xy}}\right)$

Principle stresses in 3D

- The stress tensor is a symmetric 3x3 tensor that can be written in different coordinate systems.
- From linear algebra we know that one coordinate system exists in which the tensor only has non-zero elements in its diagonal (everywhere else the components are zero).

$$\overleftrightarrow{\tau} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \xRightarrow{\text{in principal coord.}} \begin{pmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{y'x'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{z'y'} & \sigma_{z'} \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

- The axes of this coordinate system are the *principal axes*
- The elements in the diagonal are the *principal stresses*
- When the stress tensor is represented in its principal coordinate system, there are no shear stresses, only normal stresses

Principle stresses in 3D

Calculating the principal stresses

- Calculating the principal stresses equal finding the eigenvalues and eigenvectors of the stress tensor:
$$det\left(\overleftrightarrow{\sigma} - \lambda\overleftrightarrow{E}\right) = 0$$
- When we know the 3D stress state in our reference coordinate system, we can calculate the principal stresses by calculating the roots of the characteristic equation:
$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$
- With I_1, I_2, I_3:
$$\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2 \end{aligned}$$
- I_1, I_2, I_3 are the stress invariants.

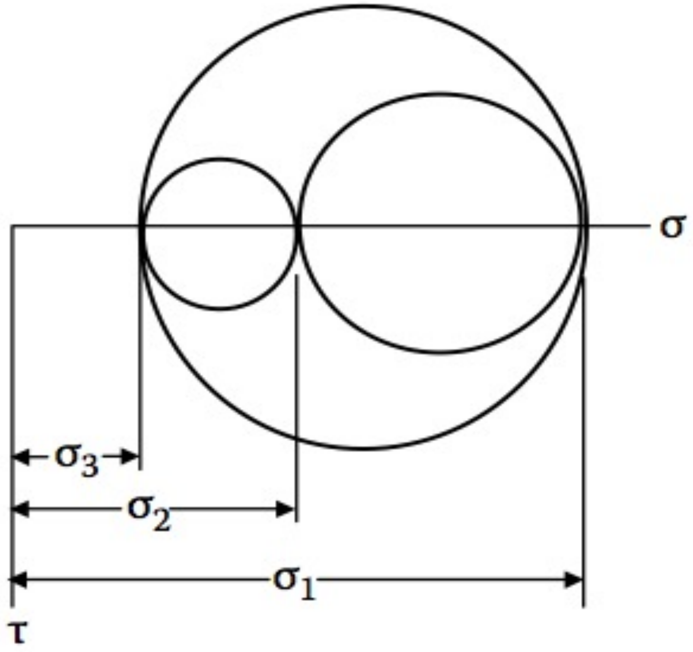
Principle stresses in 3D

- The stress invariants in the principal axes are then:

$$
\begin{aligned}
I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \\
I_3 &= \sigma_1\sigma_2\sigma_3
\end{aligned}
$$

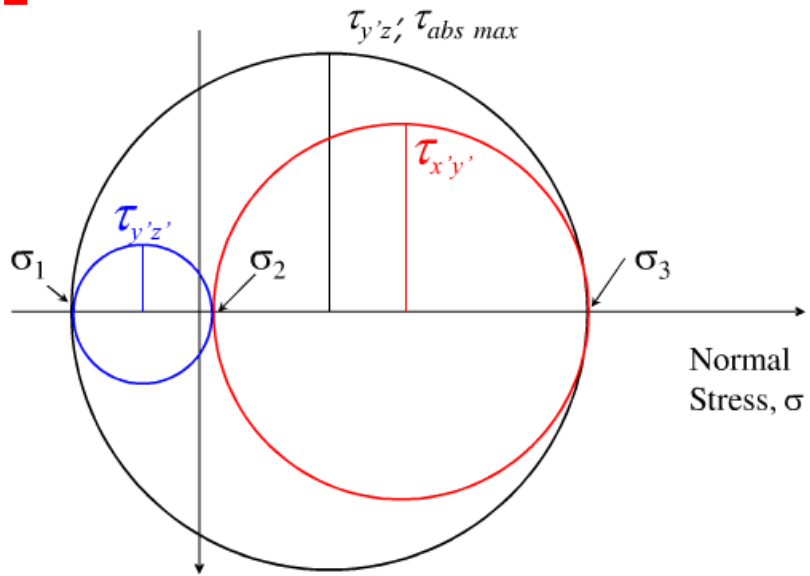
- With the *eigenvalues* of the 3D stress tensor we can then calculate the Eigenvectors. The Eigenvectors point in the direction of the principal axes of the stress state.

Mohr’s circle in 3D



- The stress tensor is dependent only on the stress state, and not on our initial choice of coordinate system.
- We’ve previously learned to draw the Mohr’s circle in 2D. Those were in essence projection of the 3D stress state in 2D
- To get to Mohr’s circle in 3D, we can therefore draw three individual Mohr’s circles for the planes x-y, x-z, and y-z, as long as we know the principal stresses

Mohr's circle in 3D- Maximum shear stress



$$\tau_{max,3} = \pm\frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{max,2} = \pm\frac{\sigma_1 - \sigma_3}{2}$$

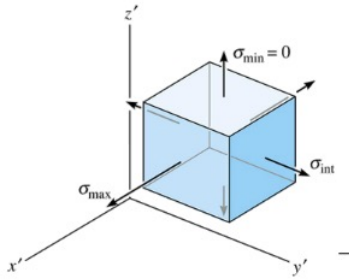
$$\tau_{max,1} = \pm\frac{\sigma_2 - \sigma_3}{2}$$

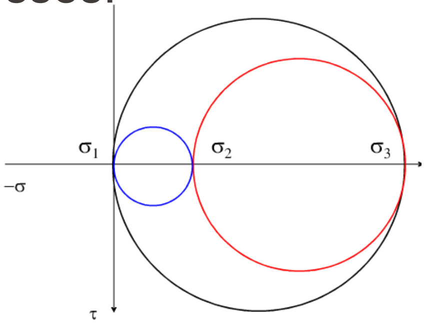
Comment: Sometimes we use the opposite numbering convention $\sigma_3<\sigma_2<\sigma_1$

We can use Mohr's circle in 3D to evaluate what the maximum shear stresses are in the 3 principal directions

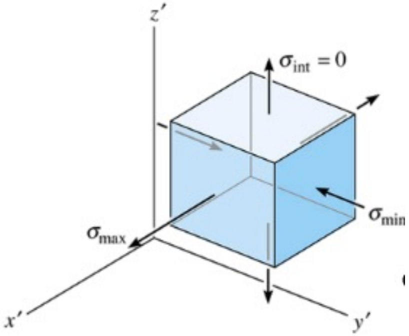
Mohr's circle in 3D - 3D state of plane stress

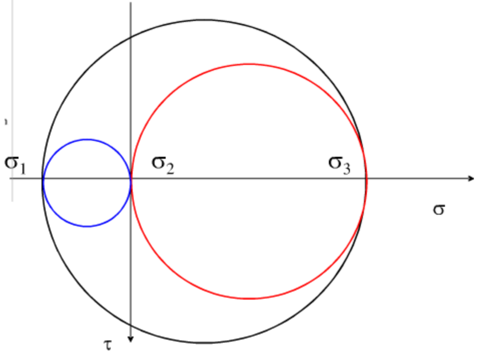
- 3D state of plane stress – 2 positive stresses:





- 3D state of plane stress – 1 positive stress, 1 negative:





Example: Triaxial stress state – NOT plane stress

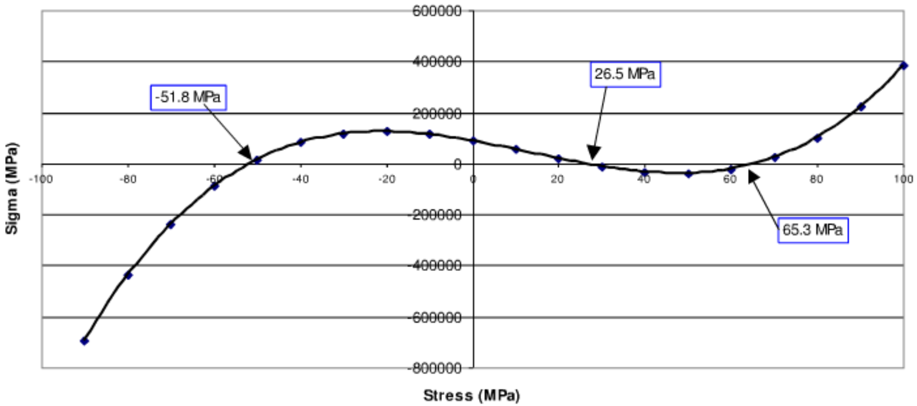
$$\overleftrightarrow{\tau} = \begin{pmatrix} 20 & 40 & -30 \\ 40 & 30 & 25 \\ -30 & 25 & -10 \end{pmatrix} MPa$$

Calculate the maximum principal stresses and maximum shear stresses for the stress state on the left.

Solution:

- Calculate stress invariants
- Calculate roots of characteristic equation (through a plot)
- Extract the maximum shear and principal stresses

Example: Triaxial stress state – NOT plane stress



$$\sigma_3 = 65.3 MPa$$

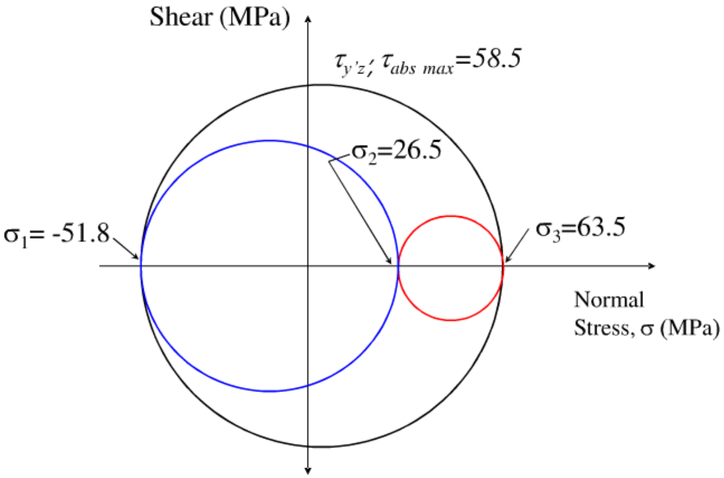
$$\sigma_2 = 26.5 MPa$$

$$\sigma_1 = -51.8 MPa$$

$$\tau_{\max} = 1/2(65.3 + 51.8)$$

$$= 58.5 MPa$$

Example: Triaxial stress state – NOT plane stress



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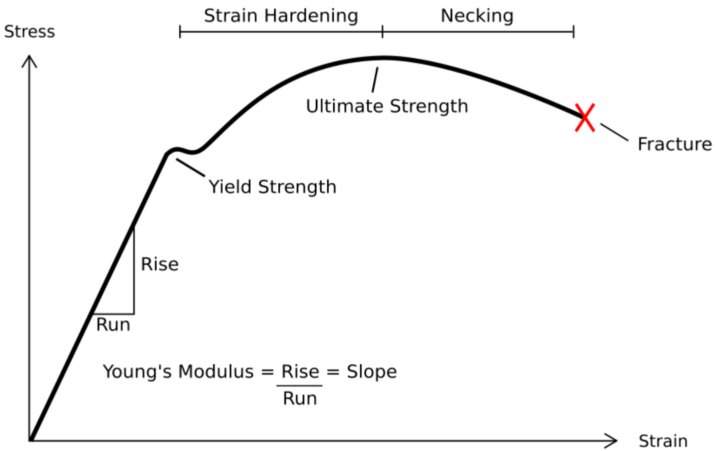
$$\sigma_1 = -51.8MPa$$

$$\tau_{\max} = 1/2(65.3 + 51.8)$$

$$= 58.5MPa$$

Failure criteria

What is Failure?



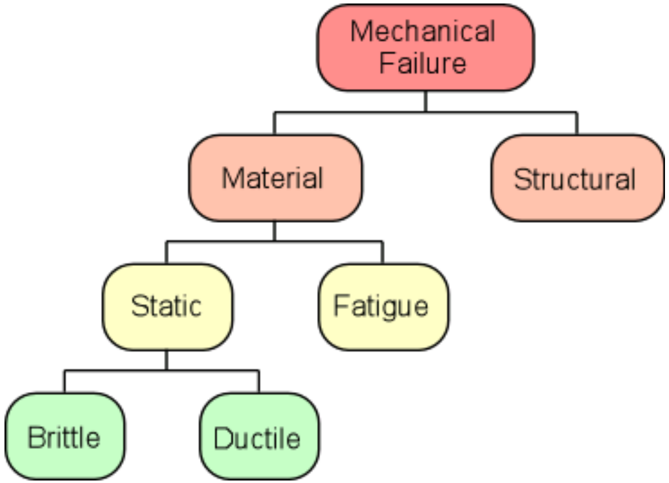
Failure – any change in a machine part which makes it unable to perform its intended function.(From Spotts M. F. and Shoup T. E.)

We will normally use a **yield failure criteria** for **ductile materials**. The ductile failure theories presented are based on yield.

Failure Theories

- Static failure
  - Ductile
  - Brittle
  - Stress concentration

- Recall
  - Ductile
    - Significant plastic deformation between yield and fracture
  - Brittle
    - Yield ~= fracture



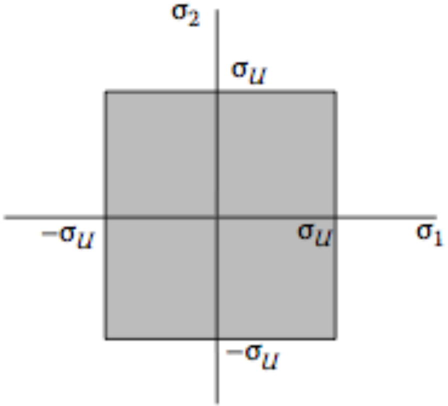
Failure of brittle materials

- A brittle material subjected to uniaxial tension fails without necking, on a plane normal to the material's long axis
- Under uniaxial tensile stress, the normal stress that causes it to fail is the ultimate tensile strength of the material
- If the material is under three-dimensional stress state, it is useful to determine the principal stresses at any given point and to use one of the failure criteria

Failure of brittle materials

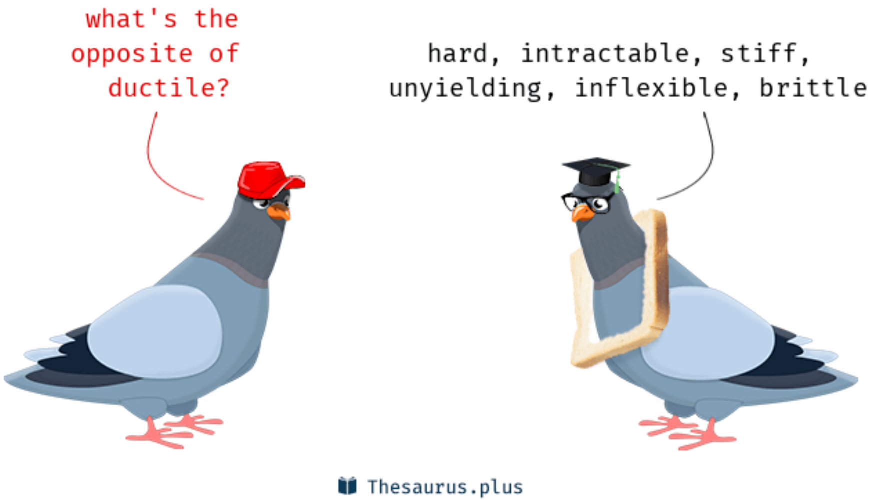
Maximum normal stress criterion

$$Max\left(|\sigma_1|, \sigma_2|, |\sigma_3|\right) = \sigma_U$$



- A given structural element fails when the maximum normal stress in that component reaches the material's ultimate tensile strength.
- This criterion should only be applied to brittle materials
- It implies that the mechanism of failure is separation
- In the case of plane stress, we can draw the maximum normal stress criterion graphically. Any state of stress within the shaded area is safe

Yield Criteria for Ductile Materials



- a ductile material subjected to uniaxial tension yields and fails by slippage along oblique surfaces and is due primarily to shear stresses
- Ductile materials fail not through fracture, but through deformation.
- plastic deformation initiated at the yield strength takes place through shear deformation, it is natural to expect failure criteria to be expressed in terms of shear stress
- We therefore cast failure criteria in terms of yield:

Von Mises criterion (distortion energy criterion)

Von Mises Criterion

- This criterion for failure of ductile materials is derived from strain energy considerations and states that yielding occurs when:

$$\boxed{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]=\sigma_Y^2}$$

- To make determining the stress state for failure analysis simpler, we can calculate an equivalent Von Mises stress for each point in the structure.

$$\boxed{\sigma_M \equiv \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}}$$

To determine whether a structural component will be safe under a given load, we should calculate the stress state at all critical points of the component and particularly at all points where stress concentrations are likely to occur.

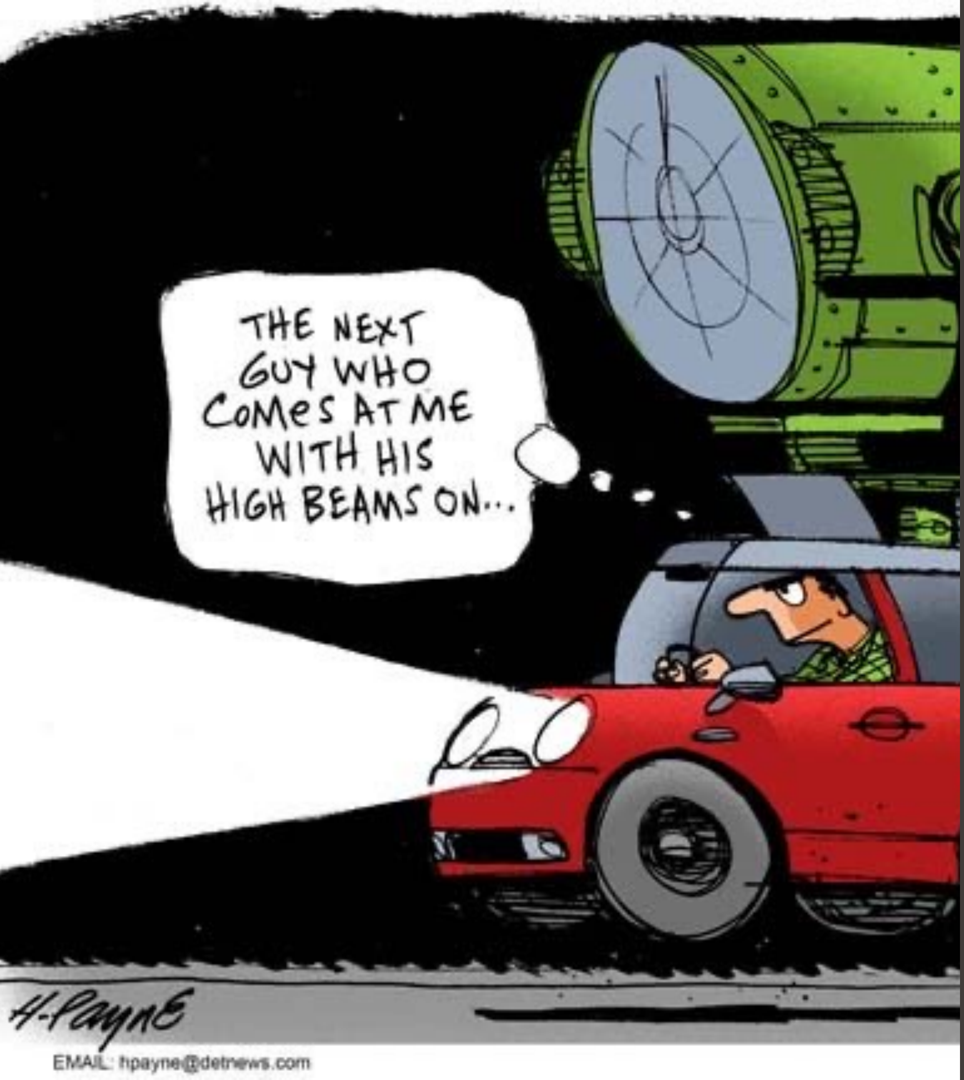
Safety factor

- We can describe how close a material in a structure is to its failure point using the safety factor.
- The safety factor compares the respective yield strength to the respective maximum or equivalent stress
- For the Von Mises safety factor we get:

$$S_M = \eta_M = \frac{\sigma_Y}{\sigma_M}$$

- sometimes the safety factor is also written as (e for equivalent):

$$\eta_e = \frac{\sigma_Y}{\sigma_e}$$



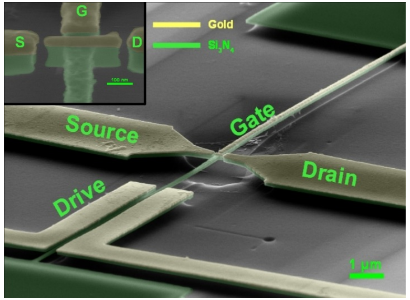
Beams

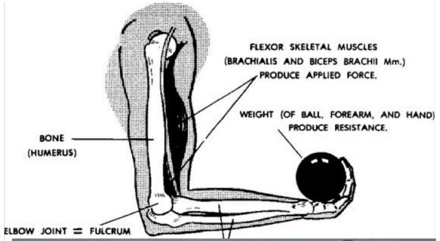
- Loads and supports
- Shear in beams
- Bending moment in beams
- Shear and moment diagrams
- Integration method for shear forces and bending moment
- Singularity functions

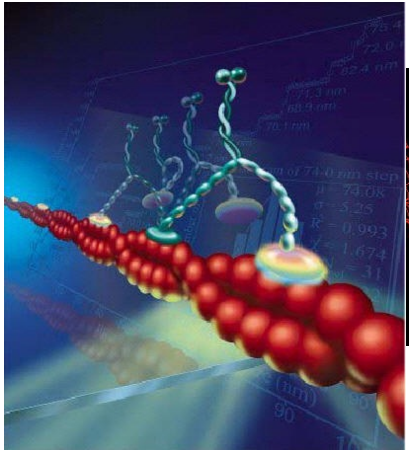
Beams

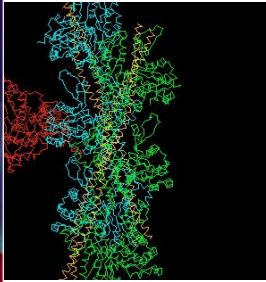
- Beams are structural members that have one dimension much longer than the other two
- A beam is a structural element that is capable of withstanding loads primarily by resisting bending.
- The bending force induced into the material of the beam as a result of the external loads, own weight, span and external reactions to these loads is called a bending moment.
- A beam with a laterally and rotationally fixed support at one end with no support at the other end is called a cantilever beam

Beams are fundamental design structures









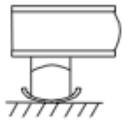
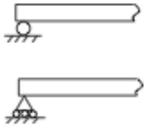
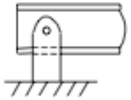
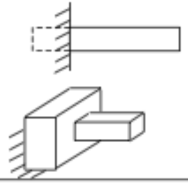
Beams

Internal reactions to external loads

- As with all previous situations, we can calculate the external reaction forces on beams through the equilibrium equations. For typical beam structures we get reaction forces and reaction moments at the supports.
- Using the method of sections we can relate the external forces and moments to internal reactions: internal shear and moments
- We can then calculate the internal shear and moments for each position of the beam and draw the shear and moment diagrams
- From these diagrams we will then in the next chapter determine how the beam deforms

Georg Fantner

Types of supports

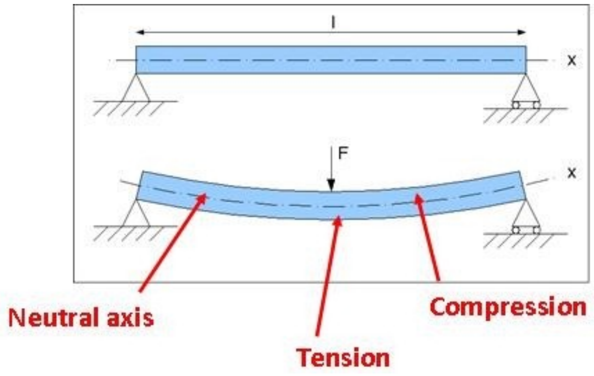
Type	Real Support	Idealized Support	Reactions Provided
Roller			
Pin or knife-edge			
Fixed			

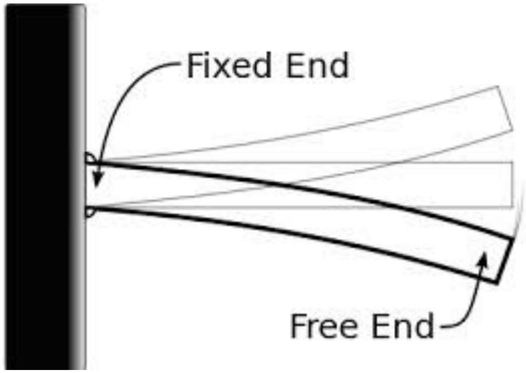
Roller or link: capable of resisting force only in one specific line of action

Pin: restricting force in any direction of the plane, so that the reaction force has two components. A pin can not withstand a moment in the plane

Fixed support: capable of resisting force in any direction as well as moments or couples

Special beams

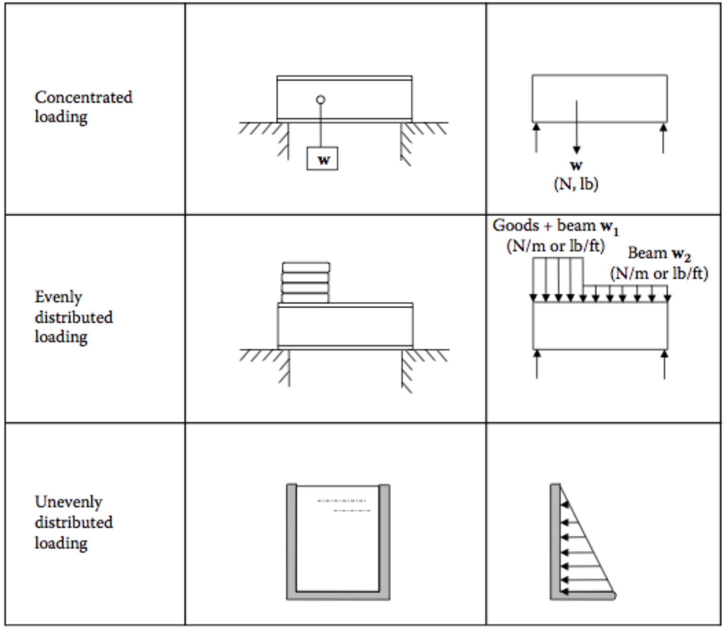




A simply supported beam is supported on one end with a pin support, on the other with a roller support.

A cantilever beam is supported on one end with a fixed support, and free on the other end

Types of loads



Concentrated loads: the force acts on a concentrated point (this is a simplification, since this is not possible in reality)

Evenly distributed loads: The force per unit area is constant. In beam problems we often look only in 2D, so then the force per unit length would be constant.

Unevenly distributed load: the force per unit area (length) varies. Often the force per unit length is given as a force intensity (q(x))

Types of problems

- Often, we can replace a distributed load by an equivalent point load acting through the centroid (center of force) of the distributed load. CAREFULL: this is only applicable for certain types or parts of calculations!
- We can separate beam problems again into statically determinate, and statically indeterminate problems. For the statically indeterminate problems we will again use constitutive laws and geometric constraints to determine the redundants

Method of Sections Applied to beams

We know from equilibrium:

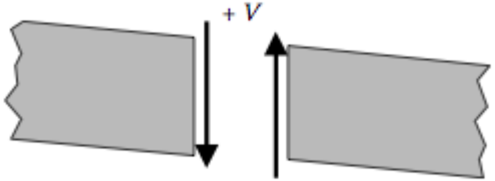
- the externally applied loads and the support reactions keep the entire body in equilibrium
- When making imaginary cuts (sections) internal reactions must exist to keep the individual sections in equilibrium. The internal reactions can be:
  - Axial force (P):
    - a horizontal force may be necessary to keep the beam in equilibrium
    - we can find axial forces by calculating
    - The line of action is always through the centroid of the beams cross-sectional area

$$\sum F_x = 0$$

Method of Sections Applied to beams

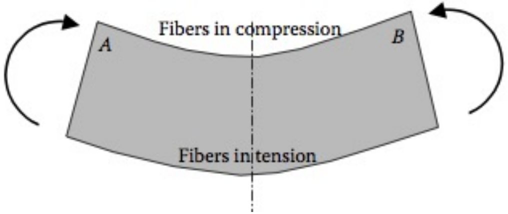
- Shear force (V):
  - A force parallel to a cut section to balance all vertical forces acting on the section
  - We find the shear forces by solving $$\sum F_z = 0$$
  - The two shear forces on two opposing faces of an imaginary cut are equal in magnitude and opposite in direction
- CONVENTION: positive shear involves downward V on the left-hand side of the cut and upward on the right.

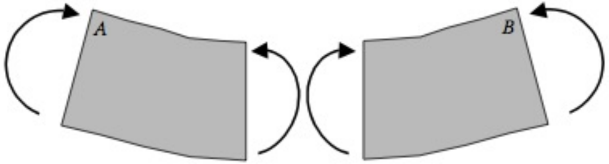
Shear in a beam is positive if the segment left of the cutting plane tends to move upwards relative to the segment to the right of the cutting plane



Method of Sections
Applied to beams

- Bending moments: these are internal moments that balance the moments that are caused by the external loads
  - the internal moment is developed within the cross-sectional area of the cut and is opposite to the resultant external moment
  - these moments tend to bend the beam: hence the name bending moment
  - the bending moment is positive when the bottom fibers are in tension, and the top fibers are in compression

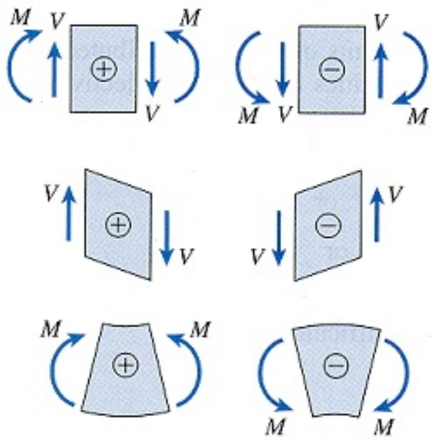


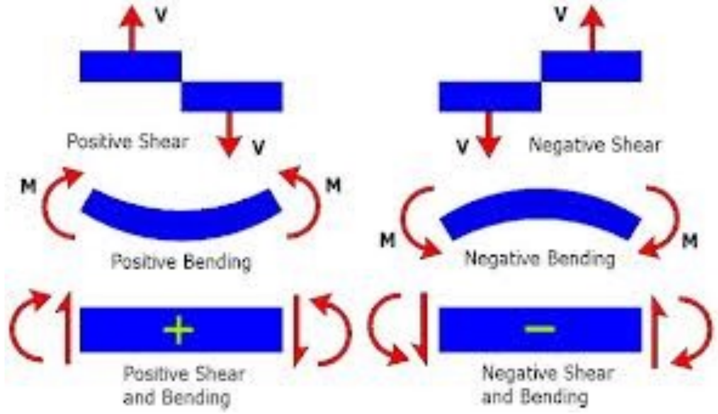


Sign convention for shear and bending moment

Shear and bending moments are resultants of stresses distributed over the cross section. The are also called: *stress resultants*

The algebraic sign of a stress resultant is determined by how it deforms the material on which it acts, NOT by its direction in space!





Beam Diagrams

- We can represent the three internal reactions in the beam each in their own diagram through the length of the beam.
  - axial force diagram
  - shear force diagram
  - bending moment diagram

- The axial force diagram is not used as often as the other two, although it can be very useful in determining the tensile stresses in flexure elements.

Beam Diagrams

- Shear Diagrams
  - The method of drawing the shear diagram is:
    1. Sketch the free body diagram
    2. find the reactions
    3. draw a V-diagram directly under the FBD
    4. find V on "points of interest" by using the method of sections and solving for F_z:
    5. Draw the V diagram and locate point of zero shear

$$\sum F_z = 0$$

Properties of shear diagrams:

- On a section where there is no external load, the shear is constant
- At a concentrated load, the shear diagram has a discontinuity
- At a uniform distributed load, the shear diagram will be a straight line with a slope equal to the load density
- For simply supported beams with vertical loads, the positive and negative areas contained by the shear diagram are equal

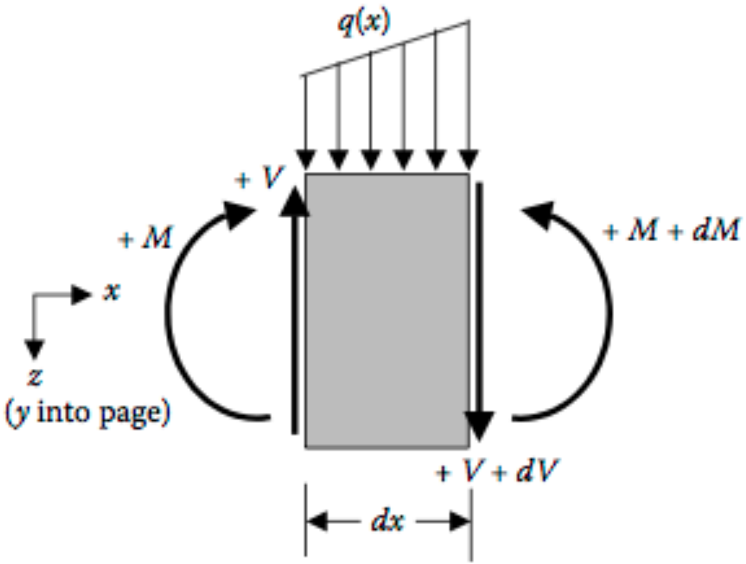
Moment Diagrams

- Drawing moment diagrams
  1. Draw M diagrams directly below shear diagrams
  2. calculate the moments at “points of interest”
     a) calculate the shear areas between key points. Add all the shear areas up beginning at the left
     b) use the FBD of individual sections beginning on the left side to compute moments at key points and points of zero shear
  3. Plot moment values: sketch shape between the plotted points by referring to the shear diagram

Properties of moment diagrams

- For simple supported, single span beams, the moment at each end is zero
- for a cantilever beam acted on by a downward force, the bending moment is zero at the free end and maximum at the support
- bending moment is positive for simply supported beams and negative for cantilever beams
- except for cantilever beams, maximum bending moment occurs at the point of zero shear

Integration method: Shear force and bending moment

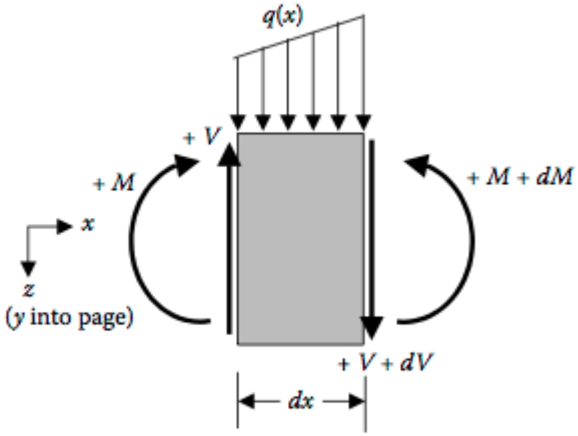


- We derive relationships between loads, shear forces and bending moments.
- We look at an infinitesimal section of the beam in bending. Let the section have length=dx
- A distributed force with intensity q(x) acts downward

Integration method: Shear force and bending moment

- From equilibrium in z:
- Eqn 1: $$\frac{dV}{dx} = -q(x)$$

- From equilibrium of moments:
- Eqn 2: $$\frac{dM}{dx} = V$$
- Combined we get:
- Eqn 3: $$\frac{d^2M}{dx^2} = -q(x)$$



Integration method: Shear force and bending moment

- From integrating eqn 1 we get:

$$V = \int dV = \int -q(x)dx + C_1$$

$$q = const \qquad V = -q \cdot x + C_1$$

This means:

- Shear is the sum of all vertical forces acting on the beam starting on the left end, up to the point of section, + the shear at the left end of the beam C_1
- Between two sections, the shear changes by the amount of vertical forces
- If a concentrated force occurs, there is a discontinuity
- for constant loads: the slope of the shear is the load density q

Integration method: Shear force and bending moment

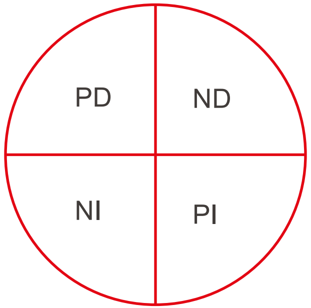
- From integrating eqn 2 we get: $$M = \int dM = \int V(x)dx + C_2$$

This means:

- the bending moment between two sections is the area under the V curve between the two sections
- C_2 we get from the boundary conditions:
  - If the beam is on rollers or pins, C_2=0
  - If the beam is on a fixed support, we can calculate the moment from the reactions

Integration method: Shear force and bending moment

- Circle to determine curvature of V and M diagram (going from q to V, or from V to M)



PD: Positive-decreasing
ND: Negative-decreasing
NI: Negative-increasing
PI: Positive-Increasing

Singularity functions

$$\langle x-a\rangle^n = \begin{cases} (x-a)^n, & \text{if } a \leq x \\ 0, & x < a \end{cases}$$

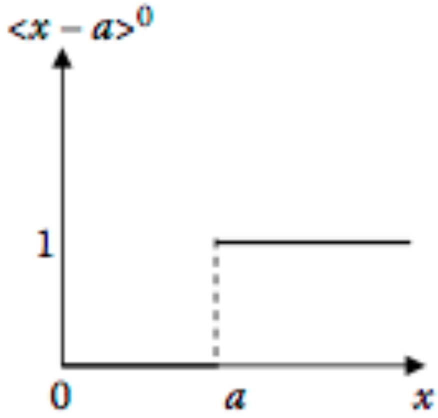
- To calculate the deflections, moments and shear diagrams of complex loading scenarios, we need a way to combine the loads into one concise formula.
- Singularity functions give us a way of adding common loading types that start acting at different distances along the beam.
- n is a positive or negative integer including 0. a is the boundary value where the load begins.
- A special case of the singularity function is for n=-1. This is the Dirac delta function and can be used to represent point loads.

Singularity functions

Standard functions n=0,1,2

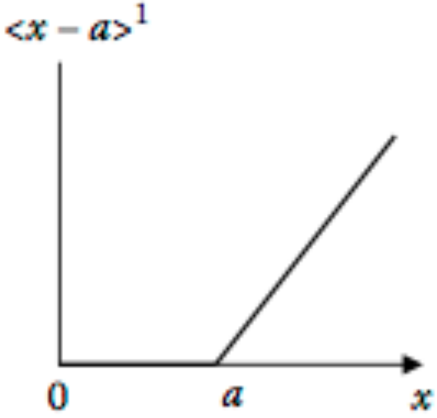
n=0

$$\langle x - x_0\rangle^0 = \begin{cases} 1 & \text{if } x \geq x_0 \\ 0 & \text{if } x < x_0 \end{cases}$$



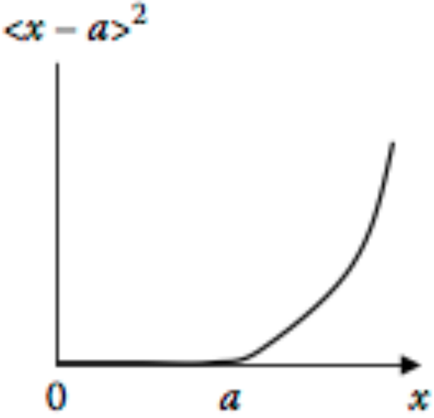
n=1

$$\langle x - x_0\rangle^1 = \begin{cases} x - a & \text{if } x \geq x_0 \\ 0 & \text{if } x < x_0 \end{cases}$$

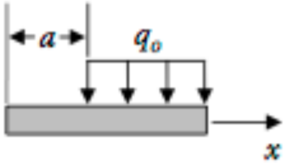


n=2

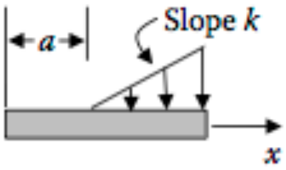
$$\langle x - x_0\rangle^2 = \begin{cases} (x - a)^2 & \text{if } x \geq x_0 \\ 0 & \text{if } x < x_0 \end{cases}$$



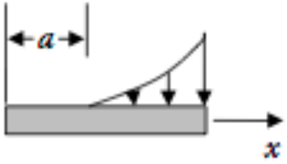
Standard loading schemes



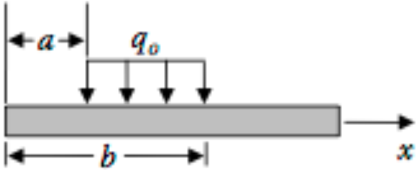
$$q(x) = q_o \langle x - a \rangle^0$$



$$q(x) = k \langle x - a \rangle^1$$



$$q(x) = k \langle x - a \rangle^2$$



$$q(x) = q_o \langle x - a \rangle^0 - q_o \langle x - b \rangle^0$$

Integration of Singularity functions

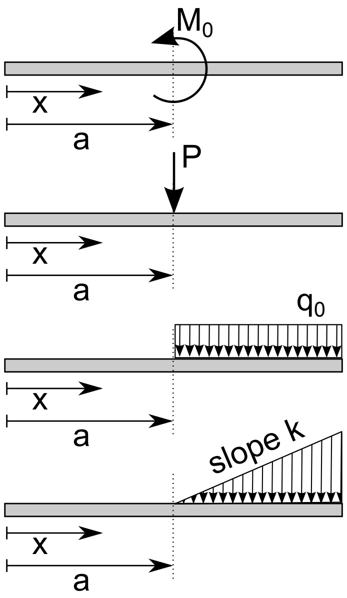
- The exponent in the singularity functions can NOT be treated as a normal exponent. It is actually an index!

$$\text{for } n \geq 0 \qquad \int \langle x-a\rangle^n dx = \frac{1}{n+1}\langle x-a\rangle^{n+1}$$

$$\text{for } n < 0 \qquad \int \langle x-a\rangle^n dx = \langle x-a\rangle^{n+1}$$

- Using the integration of the singularity functions we can now calculate the effects that standard loads q(x) have on V and M

Singularity function description of loads



	$q(x)$	$V(x)$	$M(x)$
	$M_0 \langle x-a \rangle^{-2}$	$-M_0 \langle x-a \rangle^{-1}$	$-M_0 \langle x-a \rangle^{0}$
	$P \langle x-a \rangle^{-1}$	$-P \langle x-a \rangle^{0}$	$-P \langle x-a \rangle^{1}$
	$q_0 \langle x-a \rangle^{0}$	$-q_0 \langle x-a \rangle^{1}$	$-\frac{q_0}{2} \langle x-a \rangle^{2}$
	$k \langle x-a \rangle^{1}$	$-\frac{k}{2} \langle x-a \rangle^{2}$	$-\frac{k}{6} \langle x-a \rangle^{3}$